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The Limiting Absorption Principle and Spectral Theory for Steady-State Wave Propagation in Globally Perturbed Nonselfadjoint Media*

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0. INTRODUCTION

A fundamental problem in classical physics is to describe the waves produced in a medium by the action of prescribed sources. When such sources have a sinusoidal time dependence (as is frequently postulated), the resulting waves may be expected to have the same oscillatory behavior in time, apart from a transient wave. The problem of determining this steady-state response is usually called the steady-state wave propagation problem.

This paper deals with the steady-state propagation problem for electromagnetic waves in a class of globally perturbed nonselfadjoint media. Here, we shall consider the problem in a certain subspace of inhomogeneous data.

Systems other than Maxwell's equations may be treated using the methods of this paper and such will be the subject of future work.

Up to this point integral (global) perturbations of Maxwell's equations have received little consideration, to our knowledge. Our problem is a special case of the more general problem

$$-i\partial_t u + Au + Bu = f(x, t), \tag{0.1}$$

where A is the Maxwell operator and B is a term of the form

$$(Bu)(x, t) = \int B(x, y)\, \hat{u}(y, t)\, dy. \tag{0.2}$$

A number of important results have been obtained in the case where B is a (symmetric) matrix multiplier with compact support (see [6], e.g.). Systems

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of the form (0.1) may be thought of as a generalization of the classes of symmetric hyperbolic systems introduced by Friedrichs in [3].

The steady-state problem deals with the case where the source term has the form

$$f(x,t)=e^{-i\lambda t}f(x), \qquad \lambda\in\mathbb{R}^1-\{0\} \tag{0.3}$$

and a solution of the same form is sought:

$$u(x,t)=e^{-i\lambda t}\,w(x,\lambda). \tag{0.4}$$

Thus $w(x,\lambda)$ must satisfy the equation

$$Au+Bu-\lambda u=f(x). \tag{0.5}$$

The function $w(x,\lambda)$ is not uniquely determined by Eq. (0.5) alone and it is necessary to add auxiliary conditions. Physically, a condition is needed which guarantees that $w(x,\lambda)$ behaves like an outgoing wave for $|x|\to\infty$. In our case rather simple examples exist where no such condition can be given. However, by excluding certain pathological cases, we can show the resolvent $R(\xi)=(A+B-\xi I)^{-1}$ exists as a meromorphic function in the upper and lower half planes. Hence for $\varepsilon>0$, it is possible to consider the limit

$$w(x,\lambda)=\lim_{\varepsilon\to0^+}w(x,\lambda+i\varepsilon)=\lim_{\varepsilon\to0^+}R(\lambda+i\varepsilon)f(x). \tag{0.6}$$

The limiting absorption principle states that the steady-state solution is $w(x,\lambda)$ in (0.6). Physically, $w(x,\lambda+i\varepsilon)$ is the corresponding solution in an absorptive medium and is unique when $\lambda+i\varepsilon$ is not a bonafide energy level (eigenvalue) of the system. This gives a way of defining a unique solution of (0.5) which can be thought of as "outgoing." Of course it must be proven that the limit in (0.6) exists in some appropriate sense (to be defined below). In our treatment we have borrowed certain techniques from [4, 8], a fact we greatfully acknowledge.

In Section 1 we establish the necessary background and notation. In Section 2 we prove certain facts about the resolvent set and resolvent operator. Then in Section 3 we prove that (0.6) makes sense and defines a solution to the steady-state propagation problem for certain source terms $f(x)$.

Since we are dealing with a nonnormal operator it must be expected that the spectrum of our operator has certain kinds of peculiarities. Physically, eigenvalues occurring off the real axis indicate that the system is not conservative.

It should be noted that A here is not elliptic. It has constant deficit. Such operators may have several zero speeds which may coincide in various directions. The perturbations we consider (in order to avoid some of the pathological situations already mentioned) will have roughly speaking, the same „rank" as A. This is not a very restrictive assumption as will be seen below, but it does have a somewhat nonconstructive character.

1. Background and Notation

We consider vectors in $\mathbb{C}^6$ as column vectors and write them generally as a pair of 3-vectors $f = {}^t(f^1, f^2)$, where t denotes transpose. If $u(x, t)$ is a function of $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ with values in $\mathbb{C}^6$, Maxwell's equations *in vacuo* may be written in the form of (0.2), where

$$A(D)\,u = \begin{bmatrix} i\,\mathrm{rot}\,u^2 \\ -i\,\mathrm{rot}\,u^1 \end{bmatrix} = \sum_{j=1}^{3} A_j D_j u, \tag{1.1}$$

$D_j = -i\partial_j = -i\,\partial/\partial xj$. u^1 and u^2 represent the electric and magnetic fields, respectively. The symbol $A(p)$ of $A(D)$ is a 6×6 Hermitian matrix of rank 4 when $0 \neq p = (p_1, p_2, p_3) \in \mathbb{R}^3$. We have

$$A(p) = \begin{bmatrix} 0_{3\times 3} & p_{\wedge} \\ p_{\wedge} & 0_{3\times 3} \end{bmatrix}, \tag{1.2}$$

with

$$p_{\wedge} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}. \tag{1.3}$$

The symbol $A(p)$ is of course obtained by the Fourier transform $A(p) = \Phi A(D)$, where

$$\Phi f(p) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ip\circ x} f(x)\,dx = \hat{f}(p) \qquad (f \in L^1).$$

The eigenvalues of $A(p)$ are 0, $|p|$, $-|p| = \lambda_0, \lambda_1(p), \lambda_{-1}(p)$, each of multiplicity 2 and the resolution of the identity for $A(p)$ is

$$I = P_0(p) + P_{-1}(p) + P_1(p). \tag{1.4}$$

P_i may be computed from the Dunford formula

$$P_i(x) = -(2\pi i)^{-1} \int_{|\lambda_i(x) - \xi| = \delta} [A(x) - \xi I]^{-1}\,d\xi, \tag{1.5}$$

where δ is small enough to exclude the other eigenvalues (see [12]). We know

$$A(p)\,P_i(p)=\lambda_i(p)\,P_i(p), \tag{1.6}$$

$$P_k^*(p)=P_k(p), \qquad P_k(p)\,P_j(p)=\delta_{kj}P_j(p), \tag{1.7}$$

$P_k^*(p)$ refers to the Hermitian adjoint of $P_k(p)$.

As usual $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^m)$, $\mathscr{D}(\mathbb{R}^n, \mathbb{C}^m)$ stand for the rapidly decreasing and test functions, respectively. Φ is an isomorphism on $\mathscr{S}$ which extends to $\mathscr{S}'$ by duality. By the Plancherel theorem Φ extends to $L^2(\mathbb{R}^n, \mathbb{C}^m)$(the measurable square integrable functions on $\mathbb{R}^n$ with range in $\mathbb{C}^m$). In any case we shall denote the adjoint of Φ by Φ^*. $\Phi^*f(p)=\Phi f(-p)=\overset{*}{\hat{f}}(p)$, and $\Phi\Phi^*=\Phi^*\Phi=I$. We define $BL(\mathbb{R}^3, \mathbb{C}^2)$ as the completion of $\mathscr{S}(\mathbb{R}^3, \mathbb{C}^2)$ in the norm

$$\|f\|_{BL}^2=\int_{\mathbb{R}^3}|p|^2\,|\hat{f}(p)|^2\,dp. \tag{1.8}$$

The matrices $P_0(p)$, $P_{\pm 1}(p)$ can be explicitly computed. Define $p\otimes p$ as

$$\begin{bmatrix} p_1^2 & p_1p_2 & p_1p_3 \\ p_1p_2 & p_2^2 & p_2p_3 \\ p_1p_3 & p_3p_2 & p_3^2 \end{bmatrix}. \tag{1.9}$$

Then

$$P_0(p)=\frac{1}{|p|^2}\begin{bmatrix} p\otimes p & 0_{3\times 3} \\ 0_{3\times 3} & p\otimes p \end{bmatrix}, \tag{1.10}$$

$$P_{\pm 1}(p)=\frac{1}{|p|^2}\begin{bmatrix} -(p_\times)^2 & \mp p_\times \\ \pm p_\times & -(p_\times)^2 \end{bmatrix}. \tag{1.11}$$

We define

$$P(p)=P_1(p)+P_{-1}(p). \tag{1.12}$$

$P(p)$ is an orthoprojector orthogonal to $P_0(p)$. We let $H=L^2(\mathbb{R}^3, \mathbb{C}^6)$ and define the bounded pseudodifferential operators P, P_0 on H by

$$P=\Phi^*P(p)\,\Phi, \qquad P_0=\Phi^*P_0(p)\,\Phi. \tag{1.13}$$

It is evident that P, P_0 are projections on H and we define

$$H_1=PH, \qquad H_0=P_0H. \tag{1.14}$$

Thus if $f \in H$, $f = f_1 + f_0 = Pf + P_0 f \in H_1 + H_0$. Note that $P(p)$ and $P_0(p)$ are homogeneous of order zero. Therefore, setting $w = p/|p|$, $P(w) = P(p)$, $P_0(w) = P(p)$. Define the matrix-valued functions $a(w)$, $b(w)$ as

$$\sqrt{2(w_1^2 + w_2^2)}\, a(w) = \begin{bmatrix} -w_1 w_3 & -w_2 \\ -w_2 w_3 & w_1 \\ w_1^2 + w_2^2 & 0 \end{bmatrix}, \tag{1.15}$$

$$\sqrt{2(w_1^2 + w_2^2)}\, b(w) = \begin{bmatrix} w_2 & -w_1 w_3 \\ -w_1 & -w_2 w_3 \\ 0 & w_1^2 + w_2^2 \end{bmatrix}, \tag{1.16}$$

and the map σ

$$\sigma\colon BL(\mathbb{R}^3, \mathbb{C}^2) \oplus L^2(\mathbb{R}^3, \mathbb{C}^2) \to H$$

by

$$\sigma \begin{Bmatrix} f^1 \\ f^2 \end{Bmatrix} = 2^{1/2} \Phi^*(|p|\, a(w) \hat{f}^1(p), ib(w) \hat{f}^2(p)) \tag{1.17}$$

and

$$\sigma^*\colon H \to BL(\mathbb{R}^3, \mathbb{C}^2) \oplus L^2(\mathbb{R}^3, \mathbb{C}^2)$$

by

$$\sigma^* \begin{Bmatrix} f^1 \\ f^2 \end{Bmatrix} = 2^{-1/2} \Phi^*(|p|^{-1} a^*(w)\, \hat{f}^1(p), -ib^*(w) \hat{f}^2(p)). \tag{1.18}$$

Then it is easily checked that

$$\sigma^* P = \sigma^*, \qquad P\sigma = \sigma. \tag{1.19}$$

Further, we note that

$$\Phi \sigma^* \Phi^* A(p)\, \Phi \sigma \Phi^* = i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ |p|^2 & 0 & 0 & 0 \\ 0 & |p|^2 & 0 & 0 \end{bmatrix}. \tag{1.20}$$

We define the space H_1^{loc} to be

$$\{u \in \mathscr{S}'(\mathbb{R}^3, \mathbb{C}^6) \mid \psi_A E_A u \in H \text{ for all } A;\, PE_A u = E_A u\}.$$

$E_A = \Phi^* \psi \Phi$, where $\psi \in \mathscr{D}(\mathbb{R}^3, \mathbb{C})$ such that $\operatorname{supp}(\psi) \subseteq A \Subset \mathbb{R}^3 \backslash \{0\}$, where $(\psi = \psi_A)\, A \Subset \mathbb{R}^3 \backslash \{0\}$ means A is compact in $\mathbb{R}^3 \backslash \{0\}$. It is clear that $H_1 \subset H_1^{\text{loc}}$.

For $BL(\mathbb{R}^3, \mathbb{C}^2) \oplus L^2(\mathbb{R}^3, \mathbb{C}^2)$, etc., we frequently write $BL \oplus L^2$.

2. The Resolvent

As noted above we will be interested in considering propagation through antistationary data (data from H_1). For f, u in H_1, we have

$$A(D) + PBu = f(x). \tag{2.1}$$

In general B scatters data out of H_1 and it is partly this property that makes the algebraic structure of B come into play. B may change rank "discontinuously" and we desire to eliminate this complication for the present. We will assume that,

(1) B is an integral operator with a matrix (perhaps composed with a bounded operator) kernel

$$(Bu)(x) = \int_{\mathbb{R}^3} (b_{ij}(x, y))\, \hat{u}(y)\, dy = \int_{\mathbb{R}^3} \tilde{B}(x, y)\, \hat{u}(y)\, dy. \tag{2.2}$$

(2) $\tilde{B}(x, y)$ is (uniformly) square integrable in each variable.

(3) $P\tilde{B}$ is a convolution kernel with the same rank as $A(D)$, i.e.,

$$\hat{P}(\eta)\, \Phi_\eta \tilde{B}(\cdot, y)\, \hat{u}(y) = \hat{\sigma}(\eta)\, K(\eta - y)\, \Phi_y \sigma^* u, \tag{2.3}$$

where K is a 4×4 matrix whose first and second rows are summable.

Actually assumption (3) is *almost* automatic because of the relations between $\hat{P}$ and $\hat{\sigma}$. The entries of K can be solved for by setting up certain second order equations contained in (2.3). However, we require a certain asymptotic behavior from K which restricts $\tilde{B}$ somewhat further,

$$(4)\ (1 + |x|)^{1/2 + \varepsilon}\, \check{K}(x) \in L^2(\mathbb{R}^3, \mathbb{C}^{16}). \tag{2.4}$$

Eigenfunctions of $A(D) + PB$ will have global support in general.

Lemma 2.1. *$H_1 \cap \mathscr{D}(A(D) + PB)$ is dense in H_1.*

Proof. This follows easily from Lemma 2.2 and (3) above.

LEMMA 2.2. *Let $u \in H_1 \cap \mathscr{D}(A(D))$. Then there exists an $f \in BL \oplus L^2$ such that*

$$\sigma^* PBu = (\Phi^* K) f. \tag{2.5}$$

Proof. In $\mathscr{S}'$ we have,

$$\begin{aligned}(\sigma^* PBu)(x) &= \sigma^* \hat{P}(\eta) \int \Phi_\eta B(\cdot, y)\, \hat{u}(y)\, dy \\ &= \Phi^* \int K(\eta - y)\, \hat{\sigma}_y^* u dy \\ &= (\Phi_x^* K)\, f(x),\end{aligned}$$

where f is the image of $\sigma^* u$ in $BL \oplus L^2$.

We define the operators $R_0(\xi)$ and $R(\xi)$ by

$$\sigma R_0(\xi)\, \sigma^* = (\xi I - A(D))^{-1} \tag{2.6}$$

and

$$\sigma R(\xi)\, \sigma^* = (\xi I - A(D) - PB)^{-1} \tag{2.7}$$

when the operators on the right of (2.6) and (2.7) exist as bounded operators on H_1. By 1.20 we have in $\mathscr{S}'(\mathbb{R}^3, \mathbb{C}^4)$

$$\Phi R_0(\lambda) = i \begin{bmatrix} \dfrac{\lambda}{|p|^2 - \lambda^2} & 0 & \dfrac{-1}{|p|^2 - \lambda^2} & 0 \\ 0 & \dfrac{\lambda}{|p|^2 - \lambda^2} & 0 & \dfrac{-1}{|p|^2 - \lambda^2} \\ \dfrac{|p|^2}{|p|^2 - \lambda^2} & 0 & \dfrac{\lambda}{|p|^2 - \lambda^2} & 0 \\ 0 & \dfrac{|p|^2}{|p|^2 - \lambda^2} & 0 & \dfrac{\lambda}{|p|^2 - \lambda^2} \end{bmatrix} \tag{2.8}$$

and for $\lambda \notin \mathbb{R}^1$, $R_0(\lambda)$ is a bounded operator on $BL \oplus L^2$. Since $BL \oplus L^2$ is dense in $L^2(\mathbb{R}^3, \mathbb{C}^2) \oplus L^2(\mathbb{R}^3, \mathbb{C}^2)$, we shall consider $R_0(\lambda)$ (and $R(\lambda)$) as acting in the latter space for the time being.

LEMMA 2.3. *There exist matrices $K_1(x)$, $K_2(x)$ such that*

$$\Phi_x^* K = K_1(x)\, K_2(x) = K_2(x)\, K_1(x) \tag{2.9}$$

with

$$K_2 \in L^2(\mathbb{R}^3, \mathbb{C}^{16}) \tag{2.10}$$

and

$$\sup_x \int_{\mathbb{R}^3} |x-y|^{-2} \, |K_1(y)|^2 \, dy < \infty. \tag{2.11}$$

Proof. Factor $\Phi_x^* K$ as

$$K_1(x) = (1+|x|)^{-1/2-\varepsilon} I_{4\times 4} \tag{2.12}$$

$$K_2(x) = K_1(x)^{-1} \Phi_x^* K. \tag{2.13}$$

Then (3) gives the result.

LEMMA 2.4. *There exist bounded "operators" D_1 and $D_2(\lambda)$ such that*

$$R_0(\lambda) = D_1 + D_2(\lambda) \tag{2.14}$$

and D_1, D_2 are generalized integral operators with "kernels"

$$\tilde{D}_1(x,y) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta(x-y) & 0 & 0 & 0 \\ 0 & \delta(x-y) & 0 & 0 \end{bmatrix} \tag{2.15}$$

and

$$\begin{aligned} \tilde{D}_2(x,y,\lambda) &= \tilde{D}_2^-(x,y,\lambda) \qquad \operatorname{Im}\lambda < 0, \\ &= \tilde{D}_2^+(x,y,\lambda) \qquad \operatorname{Im}\lambda > 0, \end{aligned}$$

respectively, where

$$\tilde{D}_2^\pm(x,y,\lambda) = i\sqrt{\frac{\pi}{2}} \, \frac{\exp(\pm i\lambda|x-y|)}{|x-y|} \begin{bmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 1 \\ -\lambda^2 & 0 & \lambda & 0 \\ 0 & -\lambda^2 & 0 & \lambda \end{bmatrix}. \tag{2.17}$$

Proof. The result follows immediately from taking the inverse Fourier transform of (2.8).

LEMMA 2.5. *Let* $K_i(x) = (k^i_{jl}(x))$ $i = 1, 2$, *and*

$$(s^{\pm}_{ij}(x, y, \lambda)) = K_1(x)\, \tilde{D}^{\pm}_2(x, y, \lambda)\, K_2(y). \tag{2.18}$$

Then

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |s^+_{ij}(x, y, \lambda)|^2\, dxdy \leqslant c(\lambda) &= M(\operatorname{im}\lambda)^{-1} && \operatorname{im}\lambda > 0\\ &= N \sum_{l=1}^{4} \|k^2_{il}\|^2_{L^2} \max\{|\lambda|^4, 1\} && \operatorname{im}\lambda \geqslant 0.\end{aligned} \tag{2.19}$$

a similar result holds for s^-.

Proof. Suppose $\operatorname{im}\lambda \geqslant 0$, then

$$\begin{aligned}|s^+_{ij}(x, y, \lambda)|^2 &= \left(\frac{\pi}{2}\right) \frac{\exp(-2 \operatorname{im}\lambda\, |x-y|)}{|x-y|^2}\\ &\quad \times |k^1_{ij}(x)|^2 \left|\sum_{l=1}^{4} \delta_{il}(\lambda)\, k^2_{lj}(y)\right|^2\\ &\leqslant \left(\frac{\pi}{2}\right) \frac{\exp(-2 \operatorname{im}\lambda\, |x-y|)}{|x-y|^2}\\ &\quad \times |k_{ij}(x)|^2 \left(\sum_{l=1}^{4} |\delta_{il}(\lambda)|^2\right)\left(\sum_{l=1}^{4} |k^2_{lj}(y)|^2\right),\end{aligned} \tag{*}$$

where $\delta_{il}(\lambda) = 1, 0, \lambda$, or λ^2. Equation (*) is equal to

$$\sum_{l=1}^{4} \sum_{k=1}^{4} \left(\frac{\pi}{2}\right) \frac{\exp(-2 \operatorname{im}\lambda\, |x-y|)}{|x-y|^2} |k^1_{ij}(x)|^2\, |\delta_{il}(\lambda)|^2\, |k^2_{kj}(y)|^2.$$

Now,

$$\begin{aligned}&|\delta_{il}(\lambda)|^2 \iint \frac{\pi}{2} \frac{\exp(-2 \operatorname{im}\lambda\, |x-y|)}{|x-y|^2} |k^1_{ij}(x)|^2\, |k^2_{kj}(y)|^2\, dxdy\\ &\quad\leqslant \frac{\pi}{2} \max\{|\lambda|^4, 1\} \iint \frac{\exp(-2 \operatorname{im}\lambda\, |x-y|)}{|x-y|^2 (1+|x|)^{1+2\varepsilon}}\, dx\, |k^2_{kj}(y)|^2\, dy\\ &\quad\leqslant \|k_{kj}\|_{L^2} \frac{\pi}{2} \max\{|\lambda|^4, 1\} \sup_y \int \frac{\exp(-2 \operatorname{im}\lambda\, |x-y|)}{|x-y|^2 (1+|x|)^{1+2\varepsilon}}\, dx.\end{aligned}$$

We have

$$\int \frac{\exp(-2\,\mathrm{im}\,\lambda\,|x-y|)}{|x-y|^2(1+|x|)^{1+2\varepsilon}}\,dx \leqslant \int_{\mathbb{R}^3} \frac{\exp(-2\,\mathrm{im}\,\lambda\,|w|)\,dw}{|w|^2(1+|w+y|)^{1+2\varepsilon}}$$
$$= c\int_0^\infty \frac{\exp(-2\,\mathrm{im}\,\lambda r)\,dr}{(1+|r|)^{1+2\varepsilon}}$$

and the result follows from this.

LEMMA 2.6. *Let*

$$(\Phi_x^* K) = (k_{ij}(x)).$$

Then

$$(K_1 D_1 K_2)\,f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \mathscr{K} f. \tag{2.20}$$

If $\mathscr{K}$ is bounded and 0 *is not in the essential range of*

$$(\lambda - k_{13})(\lambda - k_{24}) - k_{14}k_{23},$$

then $(\lambda I - \mathscr{K})^{-1}$ exists as a bounded operator in $L^2 \oplus L^2$.

Proof. The proof is a direct computation. A more desirable result would be obtained if some global conditions using the L^2-norm of $\mathscr{K}$ could be given (note that if $\|K\|_{L^1}$ is smaller than $|\lambda|$, then the conclusion of Lemma 2.6 holds).

A "local" condition on $\mathscr{K}$ can be given, but it is more restrictive than the hypotheses of Lemma 2.6. It is a rather curious fact that the spectrum of $\mathscr{K}\Phi$ may be controlled via the L^2-norm of $\mathscr{K}$. Note that

$$\lambda f - \mathscr{K}\hat{f} = g$$

or

$$\lambda \overset{*}{\hat{f}} - \overset{*}{\hat{\mathscr{K}}} * f = \overset{*}{\hat{g}},$$

setting $h = \overset{*}{\hat{f}}$, $k = \overset{*}{\hat{g}}$ we have

$$\lambda h - \overset{*}{\hat{\mathscr{K}}} * \hat{h} = k$$

or

$$k/\lambda - 1/\lambda \overset{*}{\hat{\mathscr{K}}} * \hat{h} = h,$$

taking $h_1 = k/\lambda$ as a first approximation, we form the series

$$\phi_1 + \phi_2 + \cdots,$$

where

$$\phi_n = h_n - h_{n-1},$$

$$h_2 = h_1 - \frac{1}{\lambda} \overset{*}{\hat{\mathscr{K}}} * \hat{h}_1,$$

$$h_n = h_1 - \frac{1}{\lambda} \overset{*}{\hat{\mathscr{K}}} * \hat{h}_{n-1},$$

by the Schwartz inequality,

$$\|\phi_n\|_\infty < |\lambda|^{-1} \|\phi_{n-1}\|_2 \|\mathscr{K}\|_2,$$

so

$$\|\overset{*}{\hat{\mathscr{K}}} * \hat{\phi}_n\|_2 = \|\mathscr{K}\phi_n\|_2 \leqslant \|\phi_n\|_\infty \|\mathscr{K}\|_2.$$

It follows that for $\|\mathscr{K}\|_2 < |\lambda|$, the series converges in $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

We denote by $s(z)$ the operator generated by the kernel $(s_{ij}(x, y, z))$ on $L^2 \oplus L^2$.

LEMMA 2.7. *Let z be fixed. Then*

$$(\lambda I - \mathscr{K} - s(z))^{-1} = (\lambda I - \mathscr{K})^{-1} + (\lambda I - \mathscr{K})^{-1} \{I - s(z)(\lambda I - \mathscr{K})^{-1}\}^{-1} \times s(z)(\lambda I - \mathscr{K})^{-1}. \tag{2.21}$$

Proof. $(\lambda I - \mathscr{K} - s(z))(\lambda I - \mathscr{K})^{-1} + (\lambda I - \mathscr{K} - s(z))(\lambda I - \mathscr{K})^{-1} \{I - s(z)(\lambda I - \mathscr{K})^{-1}\}^{-1} s(z)(\lambda I - \mathscr{K})^{-1} = I - s(z)(\lambda I - \mathscr{K})^{-1} + (I - s(z)(\lambda I - \mathscr{K})^{-1})\{I - s(z)(\lambda I - \mathscr{K})^{-1}\}^{-1} s(z)(\lambda I - \mathscr{K})^{-1} = I.$

A similar calculation shows the right-hand side of (2.21) to be the left inverse as well.

LEMMA 2.8. *$s(z)(\lambda I - \mathscr{K})^{-1}$ is an analytic operator-valued function of z in the upper and lower half-z-plane. $s(z)(\lambda I - \mathscr{K})^{-1}$ is uniformly continuous on compact subsets of the upper (lower) half-z-plane together with the real axis, extending to $\mathbb{R}^1$ via $\tilde{D}_2^+ (\tilde{D}_2^-)$.*

Proof. $s(z) = K_2 D_2(z) K_1$, and we have

$$
\begin{aligned}
\frac{s(z_1)-s(z_2)}{z_1-z_2} &= \frac{K_1 D_2(z_1) K_2 - K_1(D_2(z_2) K_2}{z_1-z_2} \\
&= \frac{K_1(D_1 + D_2(z_1) - (D_1 + D_2(z_2) K_2}{z_1-z_2} \\
&= \frac{K_1(R_0(z_1) - R_0(z_2)) K_2}{z_1-z_2} = -K_1 R_0(z_1) R_0(z_2) K_2,
\end{aligned}
$$

by the resolvent equation,

$$R_0(z_1) R_0(z_2) = (D_1 + D_1 D_2(z_1) + D_2(z_2) D_1 + D_2(z_1) D_2(z_2)),$$

so

$$
\begin{aligned}
-K_1 R_0(z_1) R_0(z_2) K_2 = -\mathscr{K} - K_1 D_1 D_2(z_1) K_2 - K_1 D_2(z_2) D_1 K_2 \\
- K_1 D_2(z_1) D_2(z_2) K_2.
\end{aligned}
$$

Since K_1 is bounded, it suffices to show that $D_2(z) K_2$ is a bounded continuous operator-valued function of z, for $\operatorname{im} z \neq 0$. It is easily checked that $\tilde{D}_2^+(z) K_2$ is a Hilbert–Schmidt kernel for $\operatorname{im} z > \varepsilon > 0$ (or $-\operatorname{im} z > \varepsilon > 0$) for all $\varepsilon > 0$. It remains to show $D_2(z) K_2$ is continuous. We may ignore the matrix portion of $D_2(z)$ for this (see (2.17)).

Since the remaining part of $\tilde{D}_2(z)$, call it $\tilde{D}_2'(z)$, is of the form $c \exp(iz\,|x-y|)/|x-y|$, we have ($|||\cdot|||$ denotes the Hilbert–Schmidt norm)

$$
\begin{aligned}
&|||D_2'(z_1) K_2 - D_2'(z_2) K_2|||^2 \\
&= \iint |K_2(x)|^2\, |\tilde{D}_2'(x, y, z_1) - \tilde{D}_2'(x, y, z_2)|^2\, dxdy \qquad (2.22) \\
&= \int_{|y|>R} \int |K_2(x)|^2\, |\tilde{D}_2'(x, y, z_1) - \tilde{D}_2'(x, y, z_2)|^2\, dxdy \\
&\quad + \int_{|y|\leqslant R} \int |K_2(x)|^2\, |\tilde{D}_2'(x, y, z_1) - \tilde{D}_2'(x, y, z_2)|^2\, dxdy \qquad (2.23) \\
&= \int_{|y|>R} \int_{|x|>R} |K_2(x)|^2\, |\tilde{D}_2'(x, y, z_1) - \tilde{D}_2'(x, y, z_2)|^2\, dxdy \\
&\quad + \int_{|y|>R} \int_{|x|\leqslant R} |K_2(x)|^2\, |\tilde{D}_2'(x, y, z_1) - \tilde{D}_2'(x, y, z_2)|^2\, dxdy \qquad (2.24) \\
&\quad + \int_{|y|\leqslant R} \int_{|x|\leqslant R} |K_2(x)|^2\, |\tilde{D}_2'(x, y, z_1) - \tilde{D}_2'(x, y, z_2)|^2\, dxdy \\
&\quad + \int_{|y|\leqslant R} \int_{|x|>R} |K_2(x)|^2\, |\tilde{D}_2'(x, y, z_1) - \tilde{D}_2'(x, y, z_2)|^2\, dxdy
\end{aligned}
$$

by the Fubini theorem, those terms in (2.24) with one integral for $|x| > R$ or $|y| > R$ may be made small uniformly in z_i. For the term with integrals over $|x| \leqslant R$ and $|y| \leqslant R$ we may use the estimate

$$|\tilde{D}_2'(x, y, z_1) - \tilde{D}_2'(x, y, z_2)| \leqslant |z_1 - z_2|. \tag{2.25}$$

This shows the required continuity.

The same type of argument also shows that $s(z)$ is uniformly continuous on compact subsets of the closed upper (lower) half plane.

LEMMA 2.10. *(See [10]).* $\{I - s(z)(\lambda I - \mathscr{K})^{-1}\}^{-1}$ *exists except for a discrete set of points in the upper (lower) half plane. When extended to the closed upper (lower) half plane, it fails to exist on the real axis in at most a nowhere dense set of linear measure zero.*

Proof. Let $z_0 \in \mathbb{R}^1$ and fix r so that if $\operatorname{im} z \geqslant 0$ and $|z - z_0| < r$, then $\|s(z_0)(\lambda I - \mathscr{K})^{-1} - s(z)(\lambda I - \mathscr{K})^{-1}\| < \frac{1}{2}$ (by Lemma 2.8). Since $s(z_0)$ is a Hilbert–Schmidt operator, so is $s(z_0)(\lambda I - \mathscr{K})^{-1}$ (see [1, p. 1012], e.g.). Therefore, there exists an operator s of finite rank such that

$$\|s(z_0)(\lambda I - \mathscr{K})^{-1} - s\| < \tfrac{1}{2}. \tag{2.26}$$

Then

$$\|s(z)(\lambda I - \mathscr{K})^{-1} - s\| < 1, \qquad |z - z_0| < r. \tag{2.27}$$

Therefore $(I - s(z)(\lambda I - \mathscr{K})^{-1} + s)^{-1}$ exists, and is analytic in the interior of $|z - z_0| < r$, $\operatorname{im} z \geqslant 0$, by its Neumann expansion.

Set

$$G(z) = s(I - s(z)(\lambda I - \mathscr{K})^{-1} + s)^{-1}. \tag{2.28}$$

We have

$$(I - s(z)(\lambda I - \mathscr{K})^{-1}) = (I - G(z))(I - s(z)(\lambda I - \mathscr{K})^{-1} + s) \tag{2.29}$$

and so there exists

$$\begin{matrix} f_1, f_2, \ldots, f_n, \\ g_1, g_2, \ldots, g_n, \end{matrix} \tag{2.30}$$

such that

$$s(g) = \sum_{i=1}^{n} (g, g_i) f_i, \tag{2.31}$$

where

$$(g, g_i) = \int_{\mathbb{R}^3} g(x)\,\overline{g_i(x)}\,dx.$$

Define

$$g_k(z) = ((I - s(z)(\lambda I - \mathscr{K})^{-1} + s)^{-1}\overset{*}{g}_k, \tag{2.32}$$

we have

$$G(z)\,g = \sum_{i=1}^{n} (g, g_i(z))\,f_i \tag{2.33}$$

and so $\{I - G(z)\}^{-1}$ exists if and only if

$$\delta(z) = \det(I_{n\times n} - ((f_i, g_j(z))_{i,j}) \neq 0 \tag{2.34}$$

(det = determinant).

It is evident that $\delta(z)$ is analytic in the interior of $\{z \mid |z - z_0| < r, \operatorname{im} z > 0\}$. $\delta(z)$ is therefore identically zero or has only a discrete set of zeros. Set $Z = \{z \mid |z - z_0| < r\}$.

Furthermore, setting $\xi = r^{-1}(z - z_0)$

$$w = (\xi^2 + i\xi + 1)(\xi^2 - i\xi + 1)^{-1}, \tag{2.35}$$

$w(z)$ takes $\{z \mid |z - z_0| < r\}$ to the unit disc in the w-plane with $(\operatorname{im} z \geqslant 0)$ $\mathbb{R}^1 \cap \{z \mid |z - z_0| < r\}$ going to the boundary of the disc. It is easily checked that a subset of $\{w \mid |w| = 1\}$ has positive circle measure if and only if its inverse image has positive measure on ∂Z. Define

$$M(w) = \delta(w^{-1}(w)) = \delta(z). \tag{2.36}$$

Appealing to Theorem 15.19 of [7], we see $\delta(z)$ vanishes on a set of measure zero in $\mathbb{R}^1$ (or else is identically zero in $\{z \mid |z - z_0| < r, \operatorname{im} z \geqslant 0\}$. Since $\delta(z)$ is continuous in $A_{r,\varepsilon} = \{z \mid z \in \mathbb{R}, |z - z_0| \leqslant r - \varepsilon, 0 < \varepsilon < r\}$, then $Z(\delta) \cap A_{r,\varepsilon}$ is closed, where $Z(\delta)$ is the set of zeros of δ.

It follows that the set of all z such that $\{I - s(z)(\lambda I - \mathscr{K})^{-1}\}^{-1}$ fails to exist is a closed set in the closed upper (lower) half-plane. Analytic continuation gives the required result taking z_0 with $\operatorname{im} z_0 > 0$.

THEOREM 2.11. *Suppose* $1 \notin \sigma(\mathscr{K})$ (cf., Lemma 2.6). *The spectrum of* $A(D) + B$ *on* H_1 *is either the entire complex plane or consists of discrete sets in the upper and lower half-planes together with the real axis.*

Proof. Suppose the spectrum is not the whole plane. Adapting Kato [5, p. 263], we see that

$$R(\lambda) = R_0(\lambda) + R_0(\lambda)\, K_2(I - \mathscr{K} - s(\lambda))^{-1}\, K_1 R_0(\lambda). \tag{2.37}$$

The right-hand side defining (uniquely) the operator $\sigma^* A(D)\,\sigma + \sigma^* PB\sigma$. In fact, it is easily seen that $(I - \mathscr{K} - s(z))^{-1}$ may be written in terms of $R(z)$ and since the operator $\sigma R(\lambda)\,\sigma^*$ is presumed to exist for some λ as a bounded operator on H_1, Lemmas 2.7–2.10 imply the result.

Remark. There are a number of conditions that can be applied to prove the resolvent set of $A(D) + B$ on H_1 is nonempty. For example, if $\| \Phi_x^* K \|_{L^{3/2}}$ is sufficiently small, then some point near the origin is in the resolvent set (eigenvalues may still be scattered throughout the plane, however). If K_2 commutes with $c(\lambda)$, where

$$c(\lambda) = \begin{bmatrix} \lambda & 0 & & 0 \\ 0 & \lambda & 0 & \\ \lambda^2 & 0 & \lambda & 0 \\ 0 & \lambda^2 & 0 & \lambda \end{bmatrix}, \tag{2.38}$$

an admitedly rather strong condition, then the spectrum does not cover the plane. Several other conditions also give the required result. In any case, the algebraic structure of B plays a strong role in determining the existence of a nonempty resolvent set and whether the steady state problem is well posed.

3. Limiting Absorption

The limiting absorption principle was used as a heuristic device in physics for many years to solve the steady-state wave propagation problem and there have been a number of rigorous justifications of the principle, (see [2, 9, e.g.).

We let S denote those points on the real axis where $(I - s(z)(I - \mathscr{K})^{-1})^{-1}$ fails to exist as a bounded operator. We assume that $1 \notin \sigma(\mathscr{K})$ and $\varepsilon > \frac{1}{2}$.

Lemma 3.1.

$$K_1 R_0(z)\, u \in H^2((a, b) \times (0, d); L^2 \oplus L^2),$$

where H^2 is the Hardy space of vector-valued functions and $-\infty < a < b < \infty$, $0 < d < \infty$ and

$$\int_a^b \|K_1 R_0(\alpha \pm i\beta)\, u\|^2\, d\alpha < c(\|u\|) < \infty \tag{3.1}$$

for $0 < \beta < d$.

Proof. Using a modification of a result of Kato [5, p. 272], it is sufficient to show that

$$\sup_{\substack{z \in (a,b) \times (0,d) \\ \|u\| = 1}} |((R_0(z) - R_0(\bar{z}))\, K_1 u, K_1 u)| < \infty. \tag{3.2}$$

However, we see that

$$\begin{aligned} &|(R(z) - R(\bar{z})\, K_1 u, K_1 u)| \\ &\quad \leqslant \iint |\tilde{D}_2^+(x, y, z) - \tilde{D}_2^-(x, y, \bar{z})|\, |(K_1 u)(y)|\, dy\, |(K_1 u)(x)|\, dx && (3.3) \\ &\quad \leqslant c(a, b, d) \iint \frac{|(K_1 u)(y)(K_1 u)(x)|}{|x - y|}\, dy dx && (3.4) \\ &\quad \leqslant c(a, b, d) \iint \left|\frac{K_1(x)\, K_1(y)}{|x - y|}\right|^2 dx dy \left(\int |u|^2\, dx\right)^2, && (3.5) \end{aligned}$$

where $c(a, b, d) = \max_{a,b,d} 2(|\lambda|^4 + 2\,|\lambda|^2 + 2)$ with $a < \operatorname{re} \lambda < b$, $0 < \operatorname{im} \lambda < d$.

By Proposition 3.1, $K_1 R_0(\alpha \pm i\beta)\, u$ has "boundary values" at almost all points on the boundary of $(a, b) \times (0, d)$, i.e.,

$$\lim_{\beta \to 0+} K_1 R_0(\alpha + i\beta)\, u = K_1 R_0(\alpha + i0)\, u \tag{3.6}$$

converges in $L^2 \oplus L^2$. Smaller values of ε might be used but this requires a more circuitous path to our goal. This may be the subject of a future investigation.

As we stated previously, we wish to prove that if

$$v(x, \alpha) = \lim_{\beta \to 0+} \sigma R(\alpha + i\beta)\, u(x), \tag{3.7}$$

then v is a (uniquely defined) solution to

$$-A(D)\, u - PBu + \lambda u = -f(x). \tag{3.8}$$

Outgoing (and incoming) solutions may be defined from this limit.

It is not appropriate to expect that the limit (3.6) should exist in H_1 since

$\mathbb{R}^1 \subseteq \sigma(A(D)+PB)$. Instead we will speak of elements in H_1^{loc}. H_1^{loc} is given the topology generated by the seminorms

$$P_A(u) = \|E_A u \psi_A\|_H. \tag{3.9}$$

THEOREM 3.3. *Let $f \in H_1$, then $f = \sigma g$, where $g \in BL \oplus L^2$. Write*

$$\begin{aligned} v_\beta = R_0(\alpha+i\beta)\,g - R_0(\alpha+i\beta)\,K_2(I-\mathscr{K}-s(\alpha+i\beta))^{-1} \\ \times K_1 R(\alpha+i\beta)\,g \end{aligned} \tag{3.10}$$

and suppose $\alpha \notin S$.

Then $\lim_{\beta\to 0} \sigma v_\beta = \sigma v_0$ *exists in H_1^{loc} and σv_0 is the unique* ("*outgoing*") *solution of the steady-state propagation problem for frequency α, and source function f.*

Proof. We know by Lemma 3.1 that $\lim_{\beta\to 0} K_1 v_\beta$ exists in $L^2 \oplus L^2$. Further, $v_\beta \in BL \oplus L^2$. v_0 is therefore a tempered distribution since $K_1^{-1}K_1 v_0 = v_0$ and $|K_1^{-1}|$ is polynomially bounded. E_A is a bounded operator on $\mathscr{S}'$ and H_1, and we have for all $\psi_A \in \mathscr{D}$, $E_A(\sigma v_0)\,\psi_A \in H_1$ since $E_A(\sigma v_0)\,\psi_A = \psi_A \sigma v_0 \overset{*}{*} \hat{\psi}_A$; $\sigma v_0 \overset{*}{*} \hat{\psi}_A$ is polynomially bounded and smooth.

Therefore $\sigma v_0 \in H_1^{\text{loc}}$. Now setting $u_\beta = \sigma v_\beta$ we have $-A(D)\,u_\beta - PBu_\beta + (\alpha+i\beta)\,u_\beta = -f \in H_1$ and letting $\beta \downarrow 0$ and utilizing (2.3) and (3.5) we have

$$-A(D)\,u_0 - PBu_0 + \alpha u_0 = -f \tag{3.11}$$

and the proof is concluded.

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